3744. [2012 : 194, 196] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers with sum 4. Prove that

$$\frac{a^8+b^8}{(a^2+b^2)^2}+\frac{b^8+c^8}{(b^2+c^2)^2}+\frac{c^8+a^8}{(c^2+a^2)^2}+abc \ge a^3+b^3+c^3.$$

Solution by Arkady Alt, San Jose, CA, USA.

Since $a+b+c=4$, the given inequality is equivalent to

$$\frac{4(a^8+b^8)}{(a^2+b^2)^2}+\frac{4(b^8+c^8)}{(b^2+c^2)^2}+\frac{4(c^8+a^8)}{(c^2+a^2)^2}+abc(a+b+c) \ge (a+b+c)(a^3+b^3+c^3). \quad (1)$$

Using the trivial inequality $2(x^2+y^2) \ge (x+y)^2$ twice, we have for $x^2+y^2 \ne 0$,

$$\frac{4(x^8+y^8)}{(x^2+y^2)^2} \ge \frac{2(x^4+y^4)^2}{(x^2+y^2)^2} \ge \frac{2(x^4+y^4)^2}{2(x^4+y^4)} = x^4+y^4.$$

Therefore,

$$\frac{4(a^8+b^8)}{(a^2+b^2)^2}+\frac{4(b^8+c^8)}{(b^2+c^2)^2}+\frac{4(c^8+a^8)}{(c^2+a^2)^2} \ge 2(a^4+b^4+c^4). \quad (2)$$

Furthermore, we have, by Schur's Inequality

$$\begin{aligned}2(a^4+b^4+c^4)&+abc(a+b+c)-(a+b+c)(a^3+b^3+c^3)\\ &= a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b) \ge 0\end{aligned}$$

so

$$2(a^4+b^4+c^4)+abc(a+b+c) \ge (a+b+c)(a^3+b^3+c^3). \quad (3)$$

Combining (2) and (3) we obtain (1) and the proof is complete.

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Cao, Geupel and Zvonaru pointed out that equality holds if and only if $a=b=c=4/3$. Magkos remarked that the given inequality actually holds for all real a, b, c as long as no two of them are simultaneously equal to zero. As usual, Wagon's solution is based on an argument using Mathematica to show that there are no counterexamples to the negation of the given inequality. He also claimed, without proof, that the inequality holds true when the sum $a+b+c=4$ is replaced by 5, 6, 7, 100, or 401/100, and that in general, the result might even be true for all a, b, and c with $a+b+c \ge 4$.